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# LAGRANGIAN INTERPOLATION

*CLARENCE ROSS*
*FLIGHT RESEARCH LABORATORY*

*SEPTEMBER 1952*

WRIGHT AIR DEVELOPMENT CENTER

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## NOTICES

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# LAGRANGIAN INTERPOLATION

*Clarence Ross*
*Flight Research Laboratory*

*September 1952*

*RDO No. 468-1*

Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

## FOREWORD

This report was prepared by Dr. Clarence Ross, Project Scientist, Computation Branch, Flight Research Laboratory. Work was completed under RDO No. 468-1, Computation Services. The opportunity to write the report was due in part to the delay in delivery of OARAC from Syracuse, New York. Because the elements tabulated were left in fraction form it was necessary to carry out a considerable amount of hand computation. Valuable assistance was afforded by Mr. Carl S. Fluke. Mr. Frank M. Williams and Rice P. White, Jr., A/1c checked most of the results.

## ABSTRACT

A systematic method of constructing formulae, together with error terms, is given for use in interpolation, extrapolation, differentiation, and integration. Both closed and open type formulae are developed, using ordinates, and based on Lagrange's interpolation formulae for equal intervals. The procedure was suggested by Professor H. H. Aiken several years ago at Harvard University. The whole procedure may be extended easily to obtain cubature formulae and formulae used for surface fitting.

## PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDING GENERAL:

Leslie B. Williams

LESLIE B. WILLIAMS
Colonel, USAF
Chief, Flight Research Laboratory

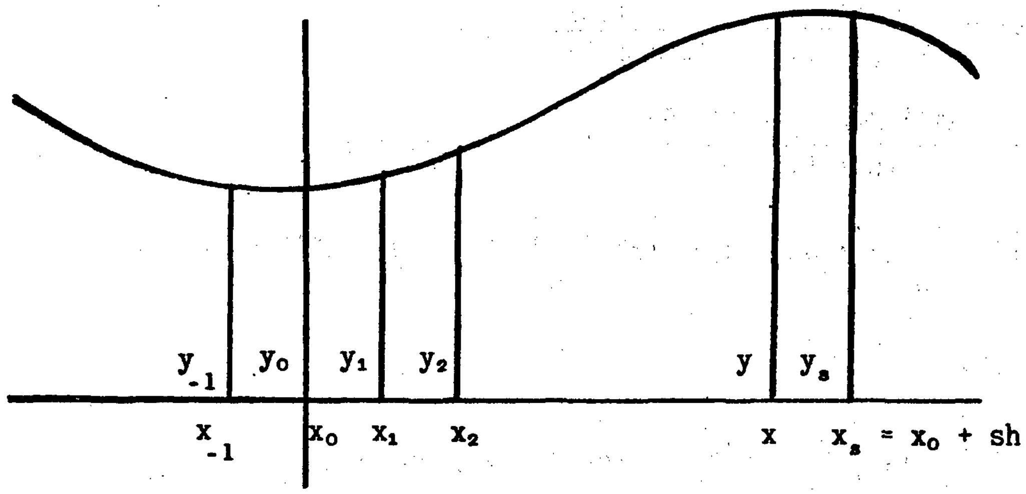
## CONTENTS

## LAGRANGIAN INTERPOLATION

The purpose of this paper is to apply Lagrange's interpclation formulae 1, 2, 3, 4 tc derive polynomial approximating formulae (using ordinates) for interpolation, differentiation, and integration. Many of these formulae appear in the literature[5] in terms of differences. Of course these differences may be expressed in terms of ordinates and in a sense the formulae which appear here are a recapitulation of elementary formulae fcund in Numerical Analysis. However, the simple derivations given here admit easy extensions to new formulae of great accuracy (exclusive of round-off error) to be used in connection with large scale high speed digital calculators. The number of ordinates employed in the formulae are indicated in the tables and include as many as ten.

### LAGRANGE'S INTERPOLATION FORMULA

The following assumptions are made and the notation is more or less standard.



(a) $x = x_0 + uh$, $y_s = y(x_0 + sh) = y(x_s)$.

(b) s, a, b are integers.

(c) $a \leq u \leq b$, where u may be integral or not.

(d) $n + 1 = b - a + 1$ is the number of points through which the approximating polynomial (1) passes.

(e) $\xi$ lies between the smallest and the largest of the set $x_s$ where $s - a = 0, 1, 2, \ldots, n$.

(f) $y^{(n+1)}(\xi)$ is familiar notation for the $(n + 1)$st derivative evaluated at $x = \xi$.

Lagrange's formula may be written

(1) $$y(x) = \sum_{s=a}^{b} C_{s-a}(x)\, y(x) + R(x) \text{ where}$$

$$C_s(x) = \frac{\phi(x)}{(x-x_s)\phi'(x_s)} = \frac{\phi(u)}{(u-s)\phi'(s)}, \text{ and}$$

(2) $$R(x) = \frac{h^{n+1}\,\phi(u)\; y^{(n+1)}(\xi)}{(n + 1)!}.$$ Also by definition

$$\phi(x) = \prod_{s=a}^{b} (x + ah - x_s) = h^{n+1} \prod_{s=0}^{n} (u - s) = h^{n+1}\,\phi(u)$$

Proof of (1).

Assume $y = \sum_{s=a}^{b} \psi_s \frac{\phi(x)}{x-x_s} + R(x)$. Then for any particular s (an integer) we have

$$y_s = \psi_s \lim_{x \to x_s} \frac{\phi(x)}{x - x_s} = \psi_s \phi'(x_s). \quad \text{Hence}$$

$$\psi_s = \frac{ys}{\phi'(x_s)} \text{ which leads to (1).}$$

Proof [6] of (2).

Write $f(z) = R(z) - R(x)\dfrac{\phi(z)}{\phi(x)}$. It follows that $f = 0$ at $z = x, x_s$ for $s - a = 0, 1, 2, 3, \ldots, n$. Hence by Rolle's theorem [7] $f^{(n+1)}(\xi) = 0$. Also it is apparent that $\phi^{(n+1)}(z) = (n + 1)!$ Consequently

$$0 = R^{(n+1)}(\xi) - R(x)\frac{(n+1)!}{\phi(x)} \text{ so that } R(x) = \frac{\phi(x)\, R^{(n+1)}(\xi)}{(n + 1)!}$$

Now by (1)

$$R^{(n+1)}(\xi) = y^{(n+1)}(\xi) \text{ which proves (2).}$$

If $b < u$ or $u < a$ then (1) may be used to extrapolate $y$ if we accept the error produced by taking $\xi$ in (2) to lie in the interval between the least and greatest of the set $x$, $x_s$ and again where $s - a = 0, 1, 2, \ldots, n$.

## DIFFERENTIATION FORMULAE

By differentiating (1) and substituting $x = x_t$ where $t$ is a particular $s$ we get

$$(3) \quad y'(x_t) = \frac{1}{h}\sum_{s=a}^{b} C'_{s-a} y_s + R'(t) \text{ where}$$

$$(4)\quad R'(t) = \frac{h^n \phi'(t)\, y^{(n+1)}(\xi)}{(n+1)!}.$$

In order to prove (4) consider (2) which may be written

$R(x) = \phi(x)\, y(x, x_s)$ since $\dfrac{y^{(n+1)}(\xi)}{(n+1)!}$ is a function of the set $x$, $x_s$

where $s - a = 0, 1, 2, \ldots, n$. Hence $R' = \phi' y + \phi y'$ and since

$\phi(x_t) = 0$ and $\phi'(x) = h^{n+1} \dfrac{d\phi}{du} \cdot \dfrac{du}{dx} = h^n \phi'(u)$ we have (4).

In order to simplify the remainder terms the derivative formulae are tabulated at the given ordinates only. Moreover not all these are tabulated since some may be implied. For example the 29 first derivative formulae imply 25 more. Indeed for $s \neq t$ we have

$$C'_s(t) = \frac{1}{\phi'_s} \lim_{u \to t} \frac{\phi(u)}{(u-t)(u-s)} = \frac{1}{t-s} \cdot \frac{\phi'_t}{\phi'_s}, \text{ and}$$

$$C'_{n-s}(n-t) = \lim_{u \to n-s} \frac{u-n+s}{\phi(u)} \cdot \lim_{u \to n-t} \frac{\phi(u)}{(u-n+t)(u-n+s)}$$

$$= \lim_{u \to n-s} \frac{u-n+s}{\phi(n-u)} \cdot \lim_{u \to n-t} \frac{\phi(n-u)}{(u-n+t)(u-n+s)}$$

$$= -\lim_{u \to s} \frac{u-s}{\phi(u)} \cdot \lim_{u \to t} \frac{\phi(u)}{(u-t)(u-s)} = -\frac{1}{t-s} \cdot \frac{\phi'_t}{\phi'_s}.$$

Hence $C'_s(t) = -C'_{n-s}(n-t)$. The diagonal elements obey this relation

also since

$$\sum_{s=0}^{n} C'_s\,(t) + \sum_{s=0}^{n} C'_{n-s}(n-t) = 0.$$

In order to obtain higher order differentiation formulae we may differentiate $y' = \frac{1}{h} C'_{s-a}\, y_s$ , omitting the remainder for a moment. Here $y'$ and $y_s$ are vectors while $C'_{s-a}$ is a square matrix of order $n + 1$. Hence in general

(5) $y^{(n)} = \frac{1}{h^n} C'^{n}_{s-a}\, y_s.$

The remainder is obtained by differentiating $R(x) = \phi(x)\, y\,(x,\, x_s)$ as before. We have in general, using symbolic notation,

(6) $R^{(n)}(x) = [\phi(x) + y(x,\, x_s)]^{(n)}.$

Only the first term of $R^{(n)}(x)$ is retained i.e. $R^{(n)}(x) = \phi^{(n)}(x)\, y(x,\, x_s)$ unless this term vanishes. If $\phi^{(n)}(x) = 0$ then obviously the order of $R^{(n)}(x)$ is multiplied by $h$ and is so indicated.

The implied differentiation formulae are obtained by using the relation $C^{(n)}_s\,(t) = (-1)^{n+1}\;\; C^{(n)}_{n-s}(n-t)$.

In the tabulation below $r^{(n)} = \dfrac{\phi^{(n)}}{(n+1)!}$.

| No. | t | LCD | $C'_0$ | $C'_1$ | $C'_2$ | $C'_3$ | $C'_4$ | $C'_5$ | $C'_6$ | $C'_7$ | $C'_8$ | $C'_9$ | $r'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 2 ordinates | | | | | | | |
| 1 | 0 | 1 | –1 | 1 | | | | | | | | | $-\frac{1}{2}$ |
| | | | | | | 3 ordinates | | | | | | | |
| 2 | 0 | 2 | –3 | 4 | –1 | | | | | | | | $\frac{1}{3}$ |
| 3 | 1 | 2 | –1 | 0 | 1 | | | | | | | | $-\frac{1}{6}$ |
| | | | | | | 4 ordinates | | | | | | | |
| 4 | 0 | 6 | –11 | 18 | –9 | 2 | | | | | | | $-\frac{1}{4}$ |
| 5 | 1 | 6 | –2 | –3 | 6 | –1 | | | | | | | $\frac{1}{12}$ |
| | | | | | | 5 ordinates | | | | | | | |
| 6 | 0 | 12 | –25 | 48 | –36 | 16 | –3 | | | | | | $\frac{1}{5}$ |
| 7 | 1 | 12 | –3 | –10 | 18 | –6 | 1 | | | | | | $-\frac{1}{20}$ |

| No. | t | LCD | $C'_0$ | $C'_1$ | $C'_2$ | $C'_3$ | $C'_4$ | $C'_5$ | $C'_6$ | $C'_7$ | $C'_8$ | $C'_9$ | $r'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 2 | 12 | 1 | −8 | 0 | 8 | −1 | | | | | | $-\frac{1}{30}$ |
| | | | | | 6 ordinates | | | | | | | | |
| 9 | 0 | 60 | −137 | 12 | −12 | 200 | −75 | 12 | | | | | $-\frac{1}{6}$ |
| 10 | 1 | 60 | −12 | −65 | 120 | −60 | 20 | −3 | | | | | $\frac{1}{30}$ |
| 11 | 2 | 60 | 3 | −30 | −20 | 60 | −15 | 2 | | | | | $-\frac{1}{60}$ |
| | | | | | 7 ordinates | | | | | | | | |
| 12 | 0 | 60 | −147 | 360 | −450 | 400 | −225 | 72 | −10 | | | | $\frac{1}{7}$ |
| 13 | 1 | 60 | −10 | 77 | 150 | −100 | 50 | −15 | 2 | | | | $-\frac{1}{42}$ |
| 14 | 2 | 60 | 2 | −24 | −35 | 80 | −30 | 8 | −1 | | | | $\frac{1}{105}$ |
| 15 | 3 | 60 | −1 | 9 | −45 | 0 | 45 | −9 | 1 | | | | $-\frac{1}{140}$ |

| No. | t | LCD | $C_0'$ | $C_1'$ | $C_2'$ | $C_3'$ | $C_4'$ | $C_5'$ | $C_6'$ | $C_7'$ | $C_8'$ | $C_9'$ | $r'$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8 ordinates | | | | | | | |
| 16 | 0 | 420 | −1089 | 2940 | −1410 | 4900 | −3675 | 1764 | −490 | 60 | | | $-\frac{1}{8}$ |
| 17 | 1 | 420 | −60 | −609 | 1260 | −1050 | 700 | −315 | 84 | −10 | | | $\frac{1}{56}$ |
| 18 | 2 | 420 | 10 | −140 | −329 | 700 | −350 | 140 | −35 | 4 | | | $-\frac{1}{168}$ |
| 19 | 3 | 420 | −4 | 42 | −252 | −105 | 420 | −126 | 28 | −3 | | | $\frac{1}{280}$ |
| | | | | | | 9 ordinates | | | | | | | |
| 20 | 0 | 840 | −2283 | 6720 | −11760 | 15680 | −14700 | 9408 | −3920 | 960 | −105 | | $\frac{1}{9}$ |
| 21 | 1 | 840 | −105 | −1338 | 2940 | −2940 | 2450 | −1470 | 588 | −140 | 15 | | $-\frac{1}{72}$ |
| 22 | 2 | 840 | 15 | −240 | −798 | 1680 | −1050 | 560 | −210 | 48 | −5 | | $\frac{1}{252}$ |
| 23 | 3 | 840 | −5 | 60 | −420 | −378 | 1050 | −420 | 140 | −30 | 3 | | $-\frac{1}{504}$ |

| No. | t | LCD | $C'_0$ | $C'_1$ | $C'_2$ | $C'_3$ | $C'_4$ | $C'_5$ | $C'_6$ | $C'_7$ | $C'_8$ | $C'_9$ | r |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 24 | 4 | 840 | 3 | −32 | 168 | −672 | 0 | 672 | −168 | 32 | −3 | | $\frac{1}{630}$ |
| 10 ordinates | | | | | | | | | | | | | |
| 25 | 0 | 2520 | −7129 | 22680 | −45360 | 70560 | −79380 | 63504 | −35280 | 12960 | −2835 | 280 | $-\frac{1}{10}$ |
| 26 | 1 | 2520 | −280 | −4329 | 10080 | −11760 | 11760 | −8820 | 4704 | −1680 | 360 | −35 | $\frac{1}{90}$ |
| 27 | 2 | 2520 | 35 | −630 | −2754 | 5880 | −4410 | 2940 | −1470 | 504 | −105 | 10 | $-\frac{1}{360}$ |
| 28 | 3 | 2520 | −10 | 135 | −1080 | −1554 | 3780 | −1890 | 840 | −270 | 54 | −5 | $\frac{1}{840}$ |
| 29 | 4 | 2520 | 5 | −60 | 360 | −1680 | −504 | 2520 | −840 | 240 | −45 | 4 | $-\frac{1}{1260}$ |

| No. | t | LCD | $C''_0$ | $C''_1$ | $C''_2$ | $C''_3$ | $C''_4$ | $C''_5$ | $C''_6$ | $C''_7$ | $C''_8$ | $C''_9$ | $r''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 3 ordinates | | | | | | | |
| 30 | 0 | 1 | 1 | −2 | 1 | | | | | | | | −1 |
| 31 | 1 | 1 | 1 | −2 | 1 | | | | | | | | $R'' = O(h^3)$ |
| | | | | | | 4 ordinates | | | | | | | |
| 32 | 0 | 1 | 2 | −5 | 4 | −1 | | | | | | | $\frac{11}{12}$ |
| 33 | 1 | 1 | 1 | −2 | 1 | 0 | | | | | | | $-\frac{1}{12}$ |
| | | | | | | 5 ordinates | | | | | | | |
| 34 | 0 | 12 | 35 | −104 | 114 | −56 | 11 | | | | | | $-\frac{5}{6}$ |
| 35 | 1 | 12 | 11 | −20 | 6 | 4 | −1 | | | | | | $\frac{1}{12}$ |
| 36 | 2 | 12 | −1 | 16 | −30 | 16 | −1 | | | | | | $R'' = O(h^5)$ |

| NO. | t | LCD | $C''_0$ | $C''_1$ | $C''_2$ | $C''_3$ | $C''_4$ | $C''_5$ | $C''_6$ | $C''_7$ | $C''_8$ | $C''_9$ | $r''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 6 ordinates | | | | | | | |
| 37 | 0 | 12 | 45 | −154 | 214 | −156 | 61 | −10 | | | | | $\frac{137}{180}$ |
| 38 | 1 | 12 | 10 | −15 | −4 | 14 | −6 | 1 | | | | | $-\frac{13}{180}$ |
| 39 | 2 | 12 | −1 | 16 | −30 | 16- | −1 | 0 | | | | | $\frac{1}{90}$ |
| | | | | | | 7 ordinates | | | | | | | |
| 40 | 0 | 180 | 812 | −3132 | 5265 | −5080 | 2970 | −972 | 137 | | | | $-\frac{7}{10}$ |
| 41 | 1 | 180 | 137 | −147 | −255 | 470 | −285 | 93 | −13 | | | | $\frac{11}{180}$ |
| 42 | 2 | 180 | −13 | 228 | −420 | 200 | 15 | −12 | 2 | | | | $-\frac{1}{90}$ |
| 43 | 3 | 180 | 2 | −27 | 270 | −490 | 270 | −27 | 2 | | | | $R'' = O(h^7)$ |

| No. | t | LCD | $C''_0$ | $C''_1$ | $C''_2$ | $C''_3$ | $C''_4$ | $C''_5$ | $C''_6$ | $C''_7$ | $C''_8$ | $C''_9$ | $r''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8 ordinates | | | | | | | |
| 44 | 0 | 180 | 938 | -4014 | 7911 | -9490 | 7380 | -3618 | 1019 | -126 | | | $\frac{363}{560}$ |
| 45 | 1 | 180 | 126 | -70 | -486 | 855 | -670 | 324 | -90 | 11 | | | $-\frac{87}{1680}$ |
| 46 | 2 | 180 | -11 | 214 | -378 | 130 | 85 | -54 | 16 | -2 | | | $\frac{47}{5040}$ |
| 47 | 3 | 180 | 2 | -27 | 270 | -490 | 270 | -27 | 2 | 0 | | | $-\frac{1}{560}$ |

| No. | t | LCD | $C''_0$ | $C''_1$ | $C''_2$ | $C''_3$ | $C''_4$ | $C''_5$ | $C''_6$ | $C''_7$ | $C''_8$ | $C''_9$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9 ordinates | | | | | | | |
| 48 | 0 | 5040 | 29531 | −138528 | 312984 | −448672 | 435330 | −284256 | 120008 | −29664 | 3267 | | $-\frac{761}{1260}$ |
| 49 | 1 | 5040 | 3267 | 128 | −20916 | 38556 | −37030 | 23688 | −9828 | 2396 | −261 | | $\frac{223}{5040}$ |
| 50 | 2 | 5040 | −261 | 5616 | −9268 | 1008 | 5670 | −4144 | 1764 | −432 | 47 | | $-\frac{19}{2520}$ |
| 51 | 3 | 5040 | 47 | −684 | 7308 | −13216 | 6930 | −252 | −196 | 72 | −9 | | $\frac{1}{560}$ |
| 52 | 4 | 5040 | −9 | 128 | −1008 | 8064 | −14350 | 8064 | −1008 | 128 | −9 | $R'' = 0(h^{9})$ | |
| | | | | | | 10 ordinates | | | | | | | |
| 53 | 0 | 5040 | 32575 | −165924 | 422568 | −704368 | 818874 | −667800 | 375704 | −139248 | 30663 | −3044 | $\frac{7129}{12600}$ |
| 54 | 1 | 5040 | 3044 | 2135 | −28944 | 57288 | −65128 | 51786 | −28560 | 10424 | −2268 | 223 | $-\frac{481}{12600}$ |
| 55 | 2 | 5040 | −223 | 5274 | −7900 | −2184 | 10458 | −8932 | 4956 | −1800 | 389 | −38 | $\frac{153}{25200}$ |
| 56 | 3 | 5040 | 38 | −603 | 6984 | −12460 | 5796 | 882 | −952 | 396 | −90 | 9 | $-\frac{37}{25200}$ |
| 57 | 4 | 5040 | −9 | 128 | −1008 | 8064 | −14350 | 8064 | −1008 | 128 | −9 | 0 | $\frac{1}{3150}$ |

| No. | t | LCD | $C_0'''$ | $C_1'''$ | $C_2'''$ | $C_3'''$ | $C_4'''$ | $C_5'''$ | $C_6'''$ | $C_7'''$ | $C_8'''$ | $C_9'''$ | $r'''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 4 ordinates | | | | | | | |
| 58 | 0 | 1 | −1 | 3 | −3 | 1 | | | | | | | $-\frac{3}{2}$ |
| 59 | 1 | 1 | −1 | 3 | −3 | 1 | | | | | | | $-\frac{1}{2}$ |
| | | | | | | 5 ordinates | | | | | | | |
| 60 | 0 | 6 | −15 | 54 | −72 | 42 | −9 | | | | | | $\frac{7}{4}$ |
| 61 | 1 | 6 | −9 | 30 | −36 | 18 | −3 | | | | | | $\frac{1}{4}$ |
| 62 | 2 | 6 | −3 | 6 | 0 | −6 | 3 | | | | | | $-\frac{1}{4}$ |
| | | | | | | 6 ordinates | | | | | | | |
| 63 | 0 | 4 | −17 | 71 | −118 | 98 | −41 | 7 | | | | | $-\frac{15}{8}$ |
| 64 | 1 | 4 | −7 | 25 | −34 | 22 | −7 | 1 | | | | | $-\frac{1}{8}$ |
| 65 | 2 | 4 | −1 | −1 | 10 | −14 | 7 | −1 | | | | | $\frac{1}{8}$ |

| No. | t | LCD | $C_0'''$ | $C_1'''$ | $C_2'''$ | $C_3'''$ | $C_4'''$ | $C_5'''$ | $C_6'''$ | $C_7'''$ | $C_8'''$ | $C_9'''$ | $r'''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7 ordinates | | | | | | | |
| 66 | 0 | 8 | −49 | 232 | −461 | 496 | −307 | 104 | −15 | | | | $\frac{29}{15}$ |
| 67 | 1 | 8 | −15 | 56 | −83 | 64 | −29 | 8 | −1 | | | | $\frac{7}{120}$ |
| 68 | 2 | 8 | −1 | −8 | 35 | −48 | 29 | −8 | 1 | | | | $-\frac{1}{15}$ |
| 69 | 3 | 8 | 1 | −8 | 13 | 0 | −13 | 8 | −1 | | | | $\frac{7}{120}$ |
| | | | | | | 8 ordinates | | | | | | | |
| 70 | 0 | 120 | −967 | 5104 | −11787 | 15560 | −12725 | 6432 | −1849 | 232 | | | $-\frac{469}{240}$ |
| 71 | 1 | 120 | −232 | 889 | −1392 | 1205 | −680 | 267 | −64 | 7 | | | $-\frac{1}{48}$ |
| 72 | 2 | 120 | −7 | −176 | 693 | −1000 | 715 | −288 | 71 | −8 | | | $\frac{3}{80}$ |
| 73 | 3 | 120 | 8 | −71 | 48 | 245 | −440 | 267 | −64 | 7 | | | $-\frac{7}{240}$ |

| No. | t | LCD | $C'''_0$ | $C'''_1$ | $C'''_2$ | $C'''_3$ | $C'''_4$ | $C'''_5$ | $C'''_6$ | $C'''_7$ | $C'''_8$ | $C'''_9$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9 ordinates | | | | | | | |
| 74 | 0 | 240 | -2403 | 13960 | -36706 | 57384 | -58280 | 39128 | -16830 | 4216 | -469 | | $\frac{29531}{15120}$ |
| 75 | 1 | 240 | -469 | 1818 | -2924 | 2690 | -1710 | 814 | -268 | 54 | -5 | | $-\frac{1}{945}$ |
| 76 | 2 | 240 | -5 | -424 | 1638 | -2504 | 2060 | -1080 | 394 | -88 | 9 | | $-\frac{331}{15120}$ |
| 77 | 3 | 240 | 9 | -86 | -100 | 882 | -1370 | 926 | -324 | 70 | -7 | | $\frac{59}{3780}$ |
| 78 | 4 | 240 | -7 | 72 | -338 | 488 | 0 | -488 | 338 | -72 | 7 | | $-\frac{41}{3024}$ |
| | | | | | | 10 ordinates | | | | | | | |
| 79 | 0 | 1512 | -180920 | 1145259 | -3375594 | 6095796 | -7392546 | 6185970 | -3540894 | 1328724 | -295326 | 29531 | $-\frac{1303}{672}$ |
| 80 | 1 | 1512 | -29531 | 114390 | -183636 | 168126 | -105714 | 49266 | -15540 | 2826 | -171 | -16 | $\frac{61}{4320}$ |
| 81 | 2 | 1512 | 16 | -29691 | 115110 | -185556 | 171486 | -109746 | 52626 | -17460 | 3546 | -331 | $\frac{79}{6048}$ |

| No. | t | LCD | $C_0'''$ | $C_1'''$ | $C_2'''$ | $C_3'''$ | $C_4'''$ | $C_5'''$ | $C_6'''$ | $C_7'''$ | $C_8'''$ | $C_9'''$ | $r'''$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 82 | 3 | 1512 | 331 | −3294 | −14796 | 75390 | −116046 | 88074 | −40236 | 12906 | −2565 | 236 | $-\frac{89}{10080}$ |
| 83 | 4 | 1512 | −236 | 2691 | −13914 | 13524 | 25830 | −56574 | 38514 | −11916 | 2286 | −205 | $\frac{41}{6048}$ |

| No. | t | LCD | $C_0^{(4)}$ | $C_1^{(4)}$ | $C_2^{(4)}$ | $C_3^{(4)}$ | $C_4^{(4)}$ | $C_5^{(4)}$ | $C_6^{(4)}$ | $C_7^{(4)}$ | $C_8^{(4)}$ | $C_9^{(4)}$ | $r^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 5 ordinates | | | | | | | |
| 84 | 0 | 1 | 1 | −4 | 6 | −4 | 1 | | | | | | −2 |
| 85 | 1 | 1 | 1 | −4 | 6 | −4 | 1 | | | | | | −1 |
| 86 | 2 | 1 | 1 | −4 | 6 | −4 | 1 | | | | | | $R^{(4)} = O(h^5)$ |
| | | | | | | 6 ordinates | | | | | | | |
| 87 | 0 | 1 | 3 | −14 | 26 | −24 | 11 | −2 | | | | | $\frac{17}{6}$ |
| 88 | 1 | 1 | 2 | −9 | 16 | −14 | 6 | −1 | | | | | $\frac{5}{6}$ |
| 89 | 2 | 1 | 1 | −4 | 6 | −4 | 1 | 0 | | | | | $-\frac{1}{6}$ |

| No. | t | LCD | $C_0^{(4)}$ | $C_1^{(4)}$ | $C_2^{(4)}$ | $C_3^{(4)}$ | $C_4^{(4)}$ | $C_5^{(4)}$ | $C_6^{(4)}$ | $C_7^{(4)}$ | $C_8^{(4)}$ | $C_9^{(4)}$ | $r^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7 ordinates | | | | | | | |
| 90 | 0 | 6 | 35 | −186 | 411 | −484 | 321 | −114 | 17 | | | | $-\frac{7}{2}$ |
| 91 | 1 | 6 | 17 | −84 | 171 | −184 | 111 | −36 | 5 | | | | $-\frac{2}{3}$ |
| 92 | 2 | 6 | 5 | −18 | 21 | −4 | −9 | 6 | −1 | | | | $\frac{1}{6}$ |
| 93 | 3 | 6 | −1 | 12 | −39 | 56 | −39 | 12 | −1 | | | $R^{(4)} = O(h^7)$ | |
| | | | | | | 8 ordinates | | | | | | | |
| 94 | 0 | 6 | 56 | −333 | 852 | −1219 | 1056 | −555 | 164 | −21 | | | $\frac{967}{240}$ |
| 95 | 1 | 6 | 21 | −112 | 255 | −324 | 251 | −120 | 33 | −4 | | | $\frac{127}{240}$ |
| 96 | 2 | 6 | 4 | −11 | 0 | 31 | −44 | 27 | −8 | 1 | | | $-\frac{11}{80}$ |
| 97 | 3 | 6 | −1 | 12 | −39 | 56 | −39 | 12 | −1 | 0 | | | $\frac{7}{240}$ |

| No. | t | LCD | $C_0^{(4)}$ | $C_1^{(4)}$ | $C_2^{(4)}$ | $C_3^{(4)}$ | $C_4^{(4)}$ | $C_5^{(4)}$ | $C_6^{(4)}$ | $C_7^{(4)}$ | $C_8^{(4)}$ | $C_9^{(4)}$ | $r^{(4)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9 ordinates | | | | | | | |
| 98 | 0 | 240 | 3207 | −21056 | 61156 | −102912 | 109930 | −76352 | 33636 | −8576 | 967 | | $-\frac{89}{20}$ |
| 99 | 1 | 240 | 967 | −5496 | 13756 | −20072 | 18930 | −11912 | 4876 | −1176 | 127 | | $-\frac{101}{240}$ |
| 100 | 2 | 240 | 127 | −176 | −924 | 3088 | −4070 | 2928 | −1244 | 304 | −33 | | $\frac{13}{120}$ |
| 101 | 3 | 240 | −33 | 424 | −1364 | 1848 | −1070 | 88 | 156 | −56 | 7 | | $-\frac{7}{240}$ |
| 102 | 4 | 240 | 7 | −96 | 676 | −1952 | 2730 | −1952 | 676 | −96 | 7 | $R^{(4)}$ | $= O(h^9)$ |
| | | | | | | 10 ordinates | | | | | | | |
| 103 | 0 | 24 | 4275 | −30668 | 99604 | −192624 | 244498 | −210920 | 123348 | −47024 | 10579 | −1068 | $\frac{4523}{945}$ |
| 104 | 1 | 24 | 1068 | −6405 | 17392 | −28556 | 31656 | −24638 | 13360 | −4812 | 1036 | −101 | $\frac{1271}{3780}$ |
| 105 | 2 | 24 | 101 | 58 | −1860 | 5272 | −7346 | 6204 | −3428 | 1240 | −267 | 26 | $-\frac{1279}{15120}$ |
| 106 | 3 | 24 | −26 | 361 | −1112 | 1260 | −188 | −794 | 744 | −308 | 70 | −7 | $\frac{359}{15120}$ |
| 107 | 4 | 24 | 7 | −96 | 676 | −1952 | 2730 | −1952 | 676 | −96 | 7 | 0 | $-\frac{41}{7560}$ |

| No. | t | LCD | $C_0^{(5)}$ | $C_1^{(5)}$ | $C_2^{(5)}$ | $C_3^{(5)}$ | $C_4^{(5)}$ | $C_5^{(5)}$ | $C_6^{(5)}$ | $C_7^{(5)}$ | $C_8^{(5)}$ | $C_9^{(5)}$ | $r^{(5)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 6 ordinates | | | | | | | |
| 108 | 0 | 1 | -1 | 5 | -10 | 10 | -5 | 1 | | | | | $-\frac{5}{2}$ |
| 109 | 1 | 1 | -1 | 5 | -10 | 10 | -5 | 1 | | | | | $-\frac{3}{2}$ |
| 110 | 2 | 1 | -1 | 5 | -10 | 10 | -5 | 1 | | | | | $-\frac{1}{2}$ |
| | | | | | | 7 ordinates | | | | | | | |
| 111 | 0 | 2 | -7 | 40 | -95 | 120 | -85 | 32 | -5 | | | | $\frac{25}{6}$ |
| 112 | 1 | 2 | -5 | 28 | -65 | 80 | -55 | 20 | -3 | | | | $\frac{5}{3}$ |
| 113 | 2 | 2 | -3 | 16 | -35 | 40 | -25 | 8 | -1 | | | | $\frac{1}{6}$ |
| 114 | 3 | 2 | -1 | 4 | -5 | 0 | 5 | -4 | 1 | | | | $-\frac{1}{3}$ |

| No. | t | LCD | $C_0^{(5)}$ | $C_1^{(5)}$ | $C_2^{(5)}$ | $C_3^{(5)}$ | $C_4^{(5)}$ | $C_5^{(5)}$ | $C_6^{(5)}$ | $C_7^{(5)}$ | $C_8^{(5)}$ | $C_9^{(5)}$ | $r^{(5)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8 ordinates | | | | | | | |
| 115 | 0 | 6 | −46 | 295 | −810 | 1235 | −1130 | 621 | −190 | 25 | | | $-\frac{35}{6}$ |
| 116 | 1 | 6 | −25 | 154 | −405 | 590 | −515 | 270 | −79 | 10 | | | $-\frac{5}{3}$ |
| 117 | 2 | 6 | −10 | 55 | −126 | 155 | −110 | 45 | −10 | 1 | | | $O(h^8)$ |
| 118 | 3 | 6 | −1 | −2 | 27 | −70 | 85 | −54 | 17 | −2 | | | $\frac{1}{6}$ |
| | | | | | | 9 ordinates | | | | | | | |
| 119 | 0 | 6 | −81 | 575 | −1790 | 3195 | −3580 | 2581 | −1170 | 305 | −35 | | $\frac{1069}{144}$ |
| 120 | 1 | 6 | −35 | 234 | −685 | 1150 | −1215 | 830 | −359 | 90 | −10 | | $\frac{229}{144}$ |
| 121 | 2 | 6 | −10 | 55 | −126 | 155 | −110 | 45 | −10 | 1 | 0 | | $-\frac{11}{144}$ |
| 122 | 3 | 6 | 0 | −10 | 55 | −126 | 155 | −110 | 45 | −10 | 1 | | $-\frac{11}{144}$ |
| 123 | 4 | 6 | 1 | −9 | 26 | −29 | 0 | 29 | −26 | 9 | −1 | | $\frac{13}{144}$ |

| No. | t | LCD | $C_0^{(5)}$ | $C_1^{(5)}$ | $C_2^{(5)}$ | $C_3^{(5)}$ | $C_4^{(5)}$ | $C_5^{(5)}$ | $C_6^{(5)}$ | $C_7^{(5)}$ | $C_8^{(5)}$ | $C_9^{(5)}$ | $r^{(5)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 10 crdinates | | | | | | | |
| 124 | 0 | 144 | -3013 | 23421 | -81444 | 166476 | -220614 | 196638 | -117876 | 45804 | -10461 | 1069 | $-\frac{285}{32}$ |
| 125 | 1 | 144 | -1069 | 7677 | -24684 | 46836 | -58014 | 48774 | -27852 | 10404 | -2301 | 229 | $-\frac{427}{288}$ |
| 126 | 2 | 144 | -229 | 1221 | -2628 | 2796 | -1254 | -306 | 684 | -372 | 99 | -11 | $\frac{31}{288}$ |
| 127 | 3 | 144 | 11 | -339 | 1716 | -3948 | 5106 | -4026 | 2004 | -636 | 123 | -11 | $\frac{1}{32}$ |
| 128 | 4 | 144 | 11 | -99 | 156 | 396 | -1638 | 2334 | -1716 | 684 | -141 | 13 | $-\frac{13}{288}$ |

| No. | t | LCD | $C_0^{(6)}$ | $C_1^{(6)}$ | $C_2^{(6)}$ | $C_3^{(6)}$ | $C_4^{(6)}$ | $C_5^{(6)}$ | $C_6^{(6)}$ | $C_7^{(6)}$ | $C_8^{(6)}$ | $C_9^{(6)}$ | $r^{(6)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7 ordinates | | | | | | | |
| 129 | 0 | 1 | 1 | -6 | 15 | -20 | 15 | -6 | 1 | | | | -3 |
| 130 | 1 | 1 | 1 | -6 | 15 | -20 | 15 | -6 | 1 | | | | -2 |

| No. | t | LCD | $C_0^{(6)}$ | $C_1^{(6)}$ | $C_2^{(6)}$ | $C_3^{(6)}$ | $C_4^{(6)}$ | $C_5^{(6)}$ | $C_6^{(6)}$ | $C_7^{(6)}$ | $C_8^{(6)}$ | $C_9^{(6)}$ | $r^{(6)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 131 | 2 | 1 | 1 | −6 | 15 | −20 | 15 | −6 | 1 | | | | −1 |
| 132 | 3 | 1 | 1 | −6 | 15 | −20 | 15 | −6 | 1 | | | $R^{(6)} = O(h^7)$ | |
| | | | | | | 8 ordinates | | | | | | | |
| 133 | 0 | 1 | 4 | −27 | 78 | −125 | 120 | −69 | 22 | −3 | | | $\frac{23}{4}$ |
| 134 | 1 | 1 | 3 | −20 | 57 | −90 | 85 | −48 | 15 | −2 | | | $\frac{11}{4}$ |
| 135 | 2 | 1 | 2 | −13 | 36 | −55 | 50 | −27 | 8 | −1 | | | $\frac{3}{4}$ |
| 136 | 3 | 1 | 1 | −6 | 15 | −20 | 15 | −6 | 1 | 0 | | | $-\frac{1}{4}$ |
| | | | | | | 9 ordinates | | | | | | | |
| 137 | 0 | 4 | 39 | −292 | 956 | −1788 | 2090 | −1564 | 732 | −196 | 23 | | −9 |
| 138 | 1 | 4 | 23 | −168 | 536 | −976 | 1110 | −808 | 368 | −96 | 11 | | $-\frac{13}{4}$ |
| 139 | 2 | 4 | 11 | −76 | 228 | −388 | 410 | −276 | 116 | −28 | 3 | | $-\frac{1}{2}$ |

| No. | t | LCD | $C_0^{(6)}$ | $C_1^{(6)}$ | $C_2^{(6)}$ | $C_3^{(6)}$ | $C_4^{(6)}$ | $C_5^{(6)}$ | $C_6^{(6)}$ | $C_7^{(6)}$ | $C_8^{(6)}$ | $C_9^{(6)}$ | $r^{(6)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 140 | 3 | 4 | 3 | –16 | 32 | –24 | –10 | 32 | –24 | 8 | –1 | | $\frac{1}{4}$ |
| 141 | 4 | 4 | –1 | 12 | –52 | 116 | –150 | 116 | –52 | 12 | –1 | $R^{(6)} = O(h^{8})$ | |
| | | | | | | | 10 ordinates | | | | | | |
| 142 | 0 | 4 | 75 | –616 | 2252 | –4812 | 6626 | –6100 | 3756 | –1492 | 347 | –36 | $\frac{3013}{240}$ |
| 143 | 1 | 4 | 36 | –285 | 1004 | –2068 | 2748 | –2446 | 1460 | –564 | 128 | –13 | $\frac{853}{240}$ |
| 144 | 2 | 4 | 13 | –94 | 300 | –556 | 662 | –528 | 284 | –100 | 21 | –2 | $\frac{73}{240}$ |
| 145 | 3 | 4 | 2 | –7 | –4 | 60 | –136 | 158 | –108 | 44 | –10 | 1 | $\frac{47}{240}$ |
| 146 | 4 | 4 | –1 | 12 | –52 | 116 | –150 | 116 | –52 | 12 | –1 | 0 | $\frac{13}{240}$ |

| No. | t | LCD | $C_0^{(7)}$ | $C_1^{(7)}$ | $C_2^{(7)}$ | $C_3^{(7)}$ | $C_4^{(7)}$ | $C_5^{(7)}$ | $C_6^{(7)}$ | $C_7^{(7)}$ | $C_8^{(7)}$ | $C_9^{(7)}$ | $r^{(7)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8 ordinates | | | | | | | |
| 147 | 0 | 1 | −1 | 7 | −21 | 35 | −35 | 21 | −7 | 1 | | | $-\frac{7}{2}$ |
| 148 | 1 | 1 | −1 | 7 | −21 | 35 | −35 | 21 | −7 | 1 | | | $-\frac{5}{2}$ |
| 149 | 2 | 1 | −1 | 7 | −21 | 35 | −35 | 21 | −7 | 1 | | | $-\frac{3}{2}$ |
| 150 | 3 | 1 | −1 | 7 | −21 | 35 | −35 | 21 | −7 | 1 | | | $-\frac{1}{2}$ |
| | | | | | | 9 ordinates | | | | | | | |
| 151 | 0 | 2 | −9 | 70 | −238 | 462 | −560 | 434 | −210 | 58 | −7 | | $\frac{91}{12}$ |
| 152 | 1 | 2 | −7 | 54 | −182 | 350 | −420 | 322 | −154 | 42 | −5 | | $\frac{49}{12}$ |
| 153 | 2 | 2 | −5 | 38 | −126 | 238 | −280 | 210 | −98 | 26 | −3 | | $\frac{19}{12}$ |
| 154 | 3 | 2 | −3 | 22 | −70 | 126 | −140 | 98 | −42 | 10 | −1 | | $\frac{1}{12}$ |
| 155 | 4 | 2 | −1 | 6 | −14 | 14 | 0 | −14 | 14 | −6 | 1 | | $-\frac{5}{12}$ |

| No. | t | LCD | $C_0^{(7)}$ | $C_1^{(7)}$ | $C_2^{(7)}$ | $C_3^{(7)}$ | $C_4^{(7)}$ | $C_5^{(7)}$ | $C_6^{(7)}$ | $C_7^{(7)}$ | $C_8^{(7)}$ | $C_9^{(7)}$ | $r^{(7)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 10 ordinates | | | | | | | |
| 156 | 0 | 12 | −145 | 1239 | −4704 | 10416 | −14826 | 14070 | −8904 | 3624 | −861 | 91 | $-\frac{105}{8}$ |
| 157 | 1 | 12 | −91 | 765 | −2856 | 6216 | −8694 | 8106 | −5040 | 2016 | −471 | 49 | $-\frac{133}{24}$ |
| 158 | 2 | 12 | −49 | 399 | −1440 | 3024 | −4074 | 3654 | −2184 | 840 | −189 | 19 | $-\frac{35}{24}$ |
| 159 | 3 | 12 | −19 | 141 | −456 | 840 | −966 | 714 | −336 | 96 | −15 | 1 | $-\frac{1}{8}$ |
| 160 | 4 | 12 | −1 | −9 | 96 | −336 | 630 | −714 | 504 | −216 | 51 | −5 | $\frac{5}{24}$ |

| No. | t | LCD | $C_0^{(8)}$ | $C_1^{(8)}$ | $C_2^{(8)}$ | $C_3^{(8)}$ | $C_4^{(8)}$ | $C_5^{(8)}$ | $C_6^{(8)}$ | $C_7^{(8)}$ | $C_8^{(8)}$ | $C_9^{(8)}$ | $r^{(8)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9 ordinates | | | | | | | |
| 161 | 0 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | | -4 |
| 162 | 1 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | | -3 |
| 163 | 2 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | | -2 |
| 164 | 3 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | | -1 |
| 165 | 4 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | $R^{(8)} = 0(h^9)$ | |
| | | | | | | 10 ordinates | | | | | | | |
| 166 | 0 | 1 | 5 | -44 | 172 | -392 | 574 | -560 | 364 | -152 | 37 | -4 | $\frac{29}{3}$ |
| 167 | 1 | 1 | 4 | -35 | 136 | -308 | 448 | -434 | 280 | -116 | 28 | -3 | $\frac{17}{3}$ |
| 168 | 2 | 1 | 3 | -26 | 100 | -224 | 322 | -308 | 196 | -80 | 19 | -2 | $\frac{8}{3}$ |
| 169 | 3 | 1 | 2 | -17 | 64 | -140 | 196 | -182 | 112 | -44 | 10 | -1 | $\frac{2}{3}$ |
| 170 | 4 | 1 | 1 | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 1 | 0 | $-\frac{1}{3}$ |

| No. | t | LCD | $C_0^{(9)}$ | $C_1^{(9)}$ | $C_2^{(9)}$ | $C_3^{(9)}$ | $C_4^{(9)}$ | $C_5^{(9)}$ | $C_6^{(9)}$ | $C_7^{(9)}$ | $C_8^{(9)}$ | $C_9^{(9)}$ | $r^{(9)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 10 ordinates | | | | | | | |
| 171 | 0 | 1 | -1 | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 1 | $-\frac{9}{2}$ |
| 172 | 1 | 1 | -1 | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 1 | $-\frac{7}{2}$ |
| 173 | 2 | 1 | -1 | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 1 | $-\frac{5}{2}$ |
| 174 | 3 | 1 | -1 | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 1 | $-\frac{3}{2}$ |
| 175 | 4 | 1 | -1 | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 1 | $-\frac{1}{2}$ |

## QUADRATURE FORMULAE

We may obtain quadrature formulae by integrating (1) to get

$$(7) \quad \int_{b-k}^{b+1} y dx = h \sum_{s=a}^{b} C^*_{s-a} \, y_s + R^* \quad \text{where}$$

$$(8) \quad R^* \leq \frac{h^{n+2} \, y^{(n+1)} \, (\xi)}{(n+1)!} \int_{-k}^{1} \phi(u+n) \, du$$

(unless the last integral vanishes) and where

$$C^*_s = \int_{-k}^{1} C_s(u+n) \, du = \frac{1}{\phi'_s} \int_{n-k}^{n+1} \frac{\phi(u) \, du}{u-s} \, .$$

In order to prove (8) consider again

$R(x) = \phi(x) \, y(x, x'_s)$ from which

$$R^* = h^{n+2} \int_{u=-k}^{1} \phi(u+n) \, y(x, x_S) \, du$$

Now $\phi(u+n) = \prod_{s=0}^{n} (u+s)$ which is monotonic increasing for $u > 0$.

By the Mean Value Theorem we may therefore write

$$\int_{0}^{1} \phi(u+n) \, y(x, x_s) \, du = \frac{y^{(n+1)} \, (\xi_1)}{(n+1)!} \int_{0}^{1} \phi(u+n) \, du \quad \text{where}$$

$0 < \xi_1 < 1$. Again by Rolle's Theorem since $\phi(u+n)$ is continuous and vanishes at $u = 0, -1$, there is a point $-1 < \eta < 0$ such that $\phi'(u+n) = 0$.

Consequently in the interval $\eta \leq u \leq 0$, $\phi(u + n)$ is monotonic increasing so that we may write

$$\int_{\eta}^{0} \phi(u + n)\, y(x, x_s)\, du = \frac{y^{(n+1)}(\xi_2)}{(n + 1)!} \int_{\eta}^{0} \phi(u + n)\, du \text{ where}$$

$\eta < \xi_2 < 0$. If $y^{(n+1)}(\xi)$ is the larger of $y^{(n+1)}(\xi_1)$, $y^{(n+1)}(\xi_2)$ then we may write

$$\int_{\eta}^{1} \phi(u + n)\, y(x, x_s)\, du \leq \frac{y^{(n+1)}(\xi)}{(n + )1!} \int_{\eta}^{1} \phi(u + n)\, du \text{ where}$$

$\eta < \xi < 1$.

Continuing for the interval $-1 \leq u < \eta$ we find that $\phi(u + n)$ is monotonic decreasing so that we may write

$$\int_{-1}^{\eta} \phi(u + n)\, y(x,x_s)\, du = \frac{y^{(n+1)}(\xi_3)}{(n + 1)!} \int_{-1}^{\eta} \phi(u + n)\, du \text{ where}$$

$-1 < \xi_3 < \eta$. Therefore we may write

$$\int_{-1}^{1} \phi(u + n)\, y(x, x_s)\, du = \frac{1}{(n+1)!} \Big[ y^{(n+1)}(\xi_3) \int_{-1}^{\eta} +$$

$$y^{(n+1)}(\xi_2) \int_{\eta}^{0} + y^{(n+1)}(\xi_1) \int_{0}^{1} \Big] \leq \frac{y^{(n+1)}(\xi)}{(n+1)!} \int_{-1}^{1} \phi(u + n)\, du$$

where $-1 < \xi < 1$ and $y^{(n+1)}(\xi)$ is the largest of $y^{(n+1)}(\xi_1)$, $y^{(n+1)}(\xi_2)$, $y^{(n+1)}(\xi_3)$. Since the polynomial $\phi$ vanishes at $-u = 0, 1, 2, \ldots, n$ we find (8) by continuing in this way.

The condition $\int_{-k}^{1} \phi(u + n)\, du \neq 0$ cannot be relaxed. However because of this condition no formula will be lost. For consider

$$\phi_{s-1,\ n-1}\, C^*_{s-1,\ n-1} = \int_{n-k-1}^{n+1-1} \frac{\phi(u)\, du}{(u-s+1)\,(u-n)} = \int_{n-k}^{n+1} \frac{\phi(u-1)\, du}{(u-s)\,(u-n-1)}$$

$$= \int_{n-k}^{n+1} \frac{\phi(u)\, du}{u(u-s)}\,.$$ Hence we have

$$\phi'_s\, C^*_s - s\, \phi'_{s-1,\ n-1}\, C^*_{s-1,\ n-1} = \int_{n-k}^{n+1} \left[ \frac{\phi(u)}{u-s} - \frac{s\, \phi(u)}{u(u-s)} \right] du$$

$$= \int_{n-k}^{n+1} \frac{\phi(u)\, du}{u} \qquad \phi'_o\, C^*_o$$

and since $s\, \phi'_{s-1,\ n-1} = \phi'_s$ we have

$$(9) \quad C^*_s = \frac{\phi'_o}{\phi'_s}\, C^*_o + C^*_{s-1,\ n-1}.$$

The $C^*_{s-1,\ n-1}$ refers to that value immediately above $C^*_{s-1}$ in the tables below. Of course

$\phi'_o\, C^*_o = \int_{-k}^{1} \phi_{n-1}\,(u + n)\, du$ where the integral is on the line immediately above $C^*_o$ in the tables. If this integral vanishes we have $C^*_s = C^*_{s-1,\ n-1}$ by (9), for $s \neq 0$, and therefore the formula is repeated.

It is easy to see that this integral cannot vanish twice in succession.

By taking 1 = 1 we obtain open quadrature formulae and by taking $1 \leq 0$ we obtain closed quadrature formulae. Because of symmetry the implied quadrature formulae double to number of formulae listed below. For example if $C_s^*$ is replaced by $C_{n-s}^*$ in a formula for extrapolating ahead we shall replace it by one which extrapolates backward. The remainders may be different because the intervals are different for $\xi$.

| No. | n+1 | $C^*_0$ | $C^*_1$ | $C^*_2$ | $C^*_3$ | $C^*_4$ | $C^*_5$ | $C^*_6$ | $C^*_7$ | $C^*_8$ | $C^*_9$ | $\int_0^1 \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1-strip | | | | | | |
| 1. | 2 | $-\frac{1}{2}$ | $\frac{3}{2}$ | | | | | | | | | $\frac{5}{6}$ |
| 2. | 3 | $\frac{5}{12}$ | $-\frac{4}{3}$ | $\frac{23}{12}$ | | | | | | | | $\frac{9}{4}$ |
| 3. | 4 | $-\frac{3}{8}$ | $\frac{37}{24}$ | $-\frac{59}{24}$ | $\frac{55}{24}$ | | | | | | | $\frac{251}{30}$ |
| 4. | 5 | $\frac{251}{720}$ | $-\frac{637}{360}$ | $\frac{109}{30}$ | $-\frac{1387}{360}$ | $\frac{1901}{720}$ | | | | | | $\frac{475}{12}$ |
| 5. | 6 | $-\frac{95}{288}$ | $\frac{959}{480}$ | $-\frac{3649}{720}$ | $\frac{4991}{720}$ | $\frac{2641}{480}$ | $\frac{4277}{1440}$ | | | | | $\frac{19087}{84}$ |
| 6. | 7 | $\frac{19087}{60480}$ | $-\frac{5603}{2520}$ | $\frac{135713}{20160}$ | $-\frac{10754}{945}$ | $\frac{235183}{20160}$ | $-\frac{18637}{2520}$ | $\frac{198721}{60480}$ | | | | $\frac{36799}{24}$ |
| 7. | 8 | $-\frac{5257}{17280}$ | $\frac{32863}{13440}$ | $-\frac{115747}{13440}$ | $\frac{2102243}{120960}$ | $-\frac{296053}{13440}$ | $\frac{242653}{13440}$ | $-\frac{1152169}{120960}$ | $\frac{16083}{4480}$ | | | $\frac{1070017}{90}$ |
| 8. | 9 | $\frac{1070017}{3628800}$ | $-\frac{4832053}{1814400}$ | $\frac{19416743}{1814400}$ | $-\frac{45586321}{1814400}$ | $\frac{862303}{22680}$ | $\frac{69927631}{1814400}$ | $\frac{47738393}{1814400}$ | $-\frac{21562603}{1814400}$ | $\frac{14097247}{3628800}$ | | $\frac{2082753}{20}$ |
| 9. | 10 | $-\frac{231417}{806400}$ | $\frac{20884811}{7257600}$ | $-\frac{2357683}{181440}$ | $\frac{15788639}{453600}$ | $-\frac{222386081}{3628800}$ | $\frac{269181919}{3628800}$ | $-\frac{28416361}{453600}$ | $\frac{6648317}{181440}$ | $-\frac{104995189}{7257600}$ | $\frac{4325321}{1036800}$ | $\frac{134211265}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-1}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 2-strip | | | | | |
| 10. | 2 | 0 | 2 | (Mid-ordinate Rule) | | | | | | | | $\frac{2}{3}$ |
| 11. | 3 | $\frac{1}{3}$ | $-\frac{2}{3}$ | $\frac{7}{3}$ | | | | | | | | 2 |
| 12. | 4 | $-\frac{1}{3}$ | $\frac{4}{3}$ | $-\frac{5}{3}$ | $\frac{8}{3}$ | | | | | | | $\frac{116}{15}$ |
| 13. | 5 | $\frac{29}{90}$ | $-\frac{73}{45}$ | $\frac{49}{15}$ | $-\frac{133}{45}$ | $\frac{269}{90}$ | | | | | | $\frac{112}{3}$ |
| 14. | 6 | $-\frac{14}{45}$ | $\frac{169}{90}$ | $-\frac{71}{15}$ | $\frac{287}{45}$ | $-\frac{203}{45}$ | $\frac{33}{10}$ | | | | | $\frac{4556}{21}$ |
| 15. | 7 | $\frac{1139}{3780}$ | $-\frac{89}{42}$ | $\frac{2687}{420}$ | $-\frac{10168}{945}$ | $\frac{4577}{420}$ | $-\frac{1327}{210}$ | $\frac{13613}{3780}$ | | | | 1476 |
| 16. | 8 | $-\frac{41}{140}$ | $\frac{2222}{945}$ | $-\frac{3473}{420}$ | $\frac{1748}{105}$ | $-\frac{79417}{3780}$ | $\frac{358}{21}$ | $-\frac{703}{84}$ | $\frac{736}{189}$ | | | $\frac{518032}{45}$ |
| 17. | 9 | $\frac{32377}{113400}$ | $-\frac{146113}{56700}$ | $\frac{293299}{28350}$ | $-\frac{1375411}{56700}$ | $\frac{415423}{11340}$ | $-\frac{2097811}{56700}$ | $\frac{709939}{28350}$ | $-\frac{604033}{56700}$ | $\frac{67711}{16200}$ | | $\frac{506368}{5}$ |
| 18. | 10 | $-\frac{3956}{14175}$ | $\frac{317209}{113400}$ | $-\frac{715777}{56700}$ | $\frac{957907}{28350}$ | $-\frac{673847}{11340}$ | $\frac{4070939}{56700}$ | $-\frac{3427027}{56700}$ | $\frac{994771}{28350}$ | $-\frac{746449}{56700}$ | $\frac{20225}{4536}$ | $\frac{32740208}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-2}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 3-strip | | | | | | |
| 19. | 3 | $\frac{3}{4}$ | 0 | $\frac{9}{4}$ | (Heun's Rule) | | | | | | | $\frac{9}{4}$ |
| 20. | 4 | $-\frac{3}{8}$ | $\frac{15}{8}$ | $-\frac{9}{8}$ | $\frac{21}{8}$ | | | | | | | $\frac{81}{10}$ |
| 21. | 5 | $\frac{27}{80}$ | $-\frac{69}{40}$ | $\frac{39}{10}$ | $-\frac{99}{40}$ | $\frac{237}{80}$ | | | | | | $\frac{153}{4}$ |
| 22. | 6 | $-\frac{51}{160}$ | $\frac{309}{160}$ | $-\frac{393}{80}$ | $\frac{567}{80}$ | $-\frac{651}{160}$ | $\frac{525}{160}$ | | | | | $\frac{6165}{28}$ |
| 23. | 7 | $\frac{137}{448}$ | $-\frac{603}{280}$ | $\frac{14601}{2240}$ | $-\frac{386}{35}$ | $\frac{26151}{2240}$ | $-\frac{1653}{280}$ | $\frac{1607}{448}$ | | | | $\frac{11925}{8}$ |
| 24. | 8 | $-\frac{265}{896}$ | $\frac{2129}{896}$ | $-\frac{37473}{4480}$ | $\frac{75577}{4480}$ | $-\frac{95783}{4480}$ | $\frac{80127}{4480}$ | $-\frac{35723}{4480}$ | $\frac{3479}{896}$ | | | $\frac{115929}{10}$ |
| 25. | 9 | $\frac{12881}{44800}$ | $-\frac{8307}{3200}$ | $\frac{233559}{22400}$ | $-\frac{548033}{22400}$ | $\frac{82872}{2240}$ | $-\frac{839583}{22400}$ | $\frac{580969}{22400}$ | $-\frac{32877}{3200}$ | $\frac{186831}{44800}$ | | $\frac{2036097}{20}$ |
| 26. | 10 | $-\frac{3591}{12800}$ | $\frac{50399}{17920}$ | $-\frac{20313}{1600}$ | $\frac{190359}{5600}$ | $-\frac{2679697}{44800}$ | $\frac{3241071}{44800}$ | $-\frac{136746}{2240}$ | $\frac{403601}{11200}$ | $-\frac{163827}{12800}$ | $\frac{398799}{89600}$ | $\frac{43830315}{44}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-3}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 4-strip | | | | | | |
| 27. | 4 | 0 | $\frac{8}{3}$ | $-\frac{4}{3}$ | $\frac{8}{3}$ | (Milne's Rule) | | | | | | $\frac{112}{15}$ |
| 28. | 5 | $\frac{14}{45}$ | $-\frac{56}{45}$ | $\frac{68}{15}$ | $-\frac{116}{45}$ | $\frac{134}{45}$ | | | | | | $\frac{112}{3}$ |
| 29. | 6 | $-\frac{14}{45}$ | $\frac{28}{15}$ | $-\frac{196}{45}$ | $\frac{344}{45}$ | $-\frac{62}{15}$ | $\frac{148}{45}$ | | | | | $\frac{4576}{21}$ |
| 30. | 7 | $\frac{286}{945}$ | $-\frac{134}{63}$ | $\frac{2018}{315}$ | $-\frac{9836}{945}$ | $\frac{3838}{315}$ | $-\frac{1874}{315}$ | $\frac{3394}{945}$ | | | | $\frac{4448}{3}$ |
| 31. | 8 | $-\frac{278}{945}$ | $\frac{248}{105}$ | $-\frac{872}{105}$ | $\frac{15784}{945}$ | $-\frac{2174}{105}$ | $\frac{1928}{105}$ | $-\frac{7568}{945}$ | $\frac{408}{105}$ | | | $\frac{520064}{45}$ |
| 32. | 9 | $\frac{4063}{14175}$ | $-\frac{36674}{14175}$ | $\frac{147244}{14175}$ | $-\frac{345248}{14175}$ | $\frac{104234}{2835}$ | $-\frac{521018}{14175}$ | $\frac{374044}{14175}$ | $-\frac{146024}{14175}$ | $\frac{59143}{14175}$ | | $\frac{508032}{5}$ |
| 33. | 10 | $-\frac{7}{25}$ | $\frac{39784}{14175}$ | $-\frac{179558}{14175}$ | $\frac{96128}{2835}$ | $-\frac{845342}{14175}$ | $\frac{1021264}{14175}$ | $-\frac{854414}{14175}$ | $\frac{516928}{14175}$ | $-\frac{181745}{14175}$ | $\frac{63112}{14175}$ | $\frac{32829952}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-4}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 5-strip | | | | | | |
| 34. | 5 | $\frac{95}{144}$ | $-\frac{25}{72}$ | $\frac{25}{6}$ | $-\frac{175}{72}$ | $\frac{425}{144}$ | | | | | | $\frac{475}{12}$ |
| 35. | 6 | $-\frac{95}{288}$ | $\frac{665}{288}$ | $-\frac{175}{48}$ | $\frac{1075}{144}$ | $-\frac{1175}{288}$ | $\frac{105}{32}$ | | | | | $\frac{18575}{84}$ |
| 36. | 7 | $\frac{3715}{12096}$ | $-\frac{365}{168}$ | $\frac{9295}{1344}$ | $-\frac{1850}{189}$ | $\frac{16225}{1344}$ | $-\frac{995}{168}$ | $\frac{43405}{12096}$ | | | | $\frac{11925}{8}$ |
| 37. | 8 | $-\frac{265}{896}$ | $\frac{57515}{24192}$ | $-\frac{22535}{2688}$ | $\frac{46415}{2688}$ | $-\frac{487225}{24192}$ | $\frac{49145}{2688}$ | $-\frac{21485}{2688}$ | $\frac{93965}{24192}$ | | | $\frac{208525}{18}$ |
| 38. | 9 | $\frac{41705}{145152}$ | $\frac{188285}{72576}$ | $\frac{756415}{72576}$ | $-\frac{1776185}{72576}$ | $\frac{169555}{4536}$ | $-\frac{2629415}{72576}$ | $\frac{1910785}{72576}$ | $-\frac{746915}{72576}$ | $\frac{605495}{145152}$ | | $\frac{406925}{4}$ |
| 39. | 10 | $-\frac{81385}{290304}$ | $\frac{815875}{290304}$ | $-\frac{460375}{36288}$ | $\frac{616375}{18144}$ | $-\frac{8679625}{145152}$ | $\frac{10553015}{145152}$ | $\frac{1084625}{18144}$ | $\frac{1321625}{36288}$ | $-\frac{3720125}{290304}$ | $\frac{184625}{41472}$ | $\frac{131410625}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-5}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 6–strip | | | | | | |
| 40. | 6 | 0 | $\frac{33}{10}$ | $-\frac{21}{5}$ | $\frac{39}{5}$ | $-\frac{21}{5}$ | $\frac{33}{10}$ | | | | | $\frac{1476}{7}$ |
| 41. | 7 | $\frac{41}{140}$ | $-\frac{123}{70}$ | $\frac{1077}{140}$ | $-\frac{352}{35}$ | $\frac{1707}{140}$ | $-\frac{417}{70}$ | $\frac{503}{140}$ | | | | 1476 |
| 42. | 8 | $-\frac{41}{140}$ | $\frac{82}{35}$ | $-\frac{1107}{140}$ | $\frac{628}{35}$ | $-\frac{2843}{140}$ | $\frac{642}{35}$ | $-\frac{1121}{140}$ | $\frac{136}{35}$ | | | $\frac{57744}{5}$ |
| 43. | 9 | $\frac{401}{1400}$ | $-\frac{1809}{700}$ | $\frac{3627}{350}$ | $-\frac{16763}{700}$ | $\frac{5319}{140}$ | $-\frac{25443}{700}$ | $\frac{9227}{350}$ | $-\frac{7209}{700}$ | $\frac{5841}{1400}$ | | $\frac{508032}{5}$ |
| 44. | 10 | $-\frac{7}{25}$ | $\frac{3929}{1400}$ | $-\frac{1773}{140}$ | $\frac{11859}{350}$ | $-\frac{41459}{700}$ | $\frac{51291}{700}$ | $-\frac{41907}{700}$ | $\frac{2551}{70}$ | $-\frac{8973}{700}$ | $\frac{6233}{1400}$ | $\frac{10944720}{11}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-6}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7-strip | | | | | | |
| 45. | 7 | $\frac{5257}{8640}$ | $-\frac{49}{72}$ | $\frac{19943}{2880}$ | $-\frac{1274}{135}$ | $\frac{34153}{2880}$ | $-\frac{2107}{360}$ | $\frac{30919}{8640}$ | | | | $\frac{36799}{24}$ |
| 46. | 8 | $-\frac{5257}{17280}$ | $\frac{5257}{1920}$ | $-\frac{13573}{1920}$ | $\frac{303653}{17280}$ | $-\frac{38563}{1920}$ | $\frac{35035}{1920}$ | $-\frac{137935}{17280}$ | $\frac{2485}{640}$ | | | $\frac{1046689}{90}$ |
| 47. | 9 | $\frac{149527}{518400}$ | $-\frac{676963}{259200}$ | $\frac{2803073}{259200}$ | $-\frac{6019111}{259200}$ | $\frac{122353}{3240}$ | $-\frac{9392761}{259200}$ | $\frac{6823103}{259200}$ | $\frac{2667133}{259200}$ | $\frac{2162377}{518400}$ | | $\frac{2036097}{20}$ |
| 48. | 10 | $-\frac{3591}{12800}$ | $\frac{2916893}{1036800}$ | $-\frac{1647401}{129600}$ | $\frac{2227841}{64800}$ | $-\frac{6072619}{103680}$ | $\frac{37901353}{518400}$ | $-\frac{3875263}{64800}$ | $\frac{4720471}{129600}$ | $-\frac{13286371}{1036800}$ | $\frac{184625}{41472}$ | $\frac{131427457}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-7}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8-strip | | | | | | |
| 49. | 8 | 0 | $\frac{736}{189}$ | $-\frac{2544}{315}$ | $\frac{1952}{105}$ | $-\frac{19672}{945}$ | $\frac{1952}{105}$ | $-\frac{2544}{315}$ | $\frac{736}{189}$ | | | $\frac{506368}{45}$ |
| 50. | 9 | $\frac{3956}{14175}$ | $-\frac{31648}{14175}$ | $\frac{165968}{14175}$ | $-\frac{336016}{14175}$ | $\frac{108088}{2835}$ | $-\frac{516616}{14175}$ | $\frac{374288}{14175}$ | $-\frac{146128}{14175}$ | $\frac{59156}{14175}$ | | $\frac{506368}{5}$ |
| 51. | 10 | $-\frac{3956}{14175}$ | $\frac{7912}{2835}$ | $-\frac{174064}{14175}$ | $\frac{498272}{14175}$ | $-\frac{834472}{14175}$ | $\frac{1038896}{14175}$ | $-\frac{169784}{2835}$ | $\frac{516704}{14175}$ | $-\frac{181732}{14175}$ | $\frac{9016}{2025}$ | $\frac{32814080}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-8}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9-strip | | | | | | |
| 52. | 9 | $\frac{25713}{44800}$ | $-\frac{22437}{22400}$ | $\frac{233847}{22400}$ | $-\frac{496449}{22400}$ | $\frac{10287}{280}$ | $-\frac{796959}{22400}$ | $\frac{583497}{22400}$ | $-\frac{228987}{22400}$ | $\frac{26649}{6400}$ | | $\frac{2082753}{20}$ |
| 53. | 10 | $-\frac{25713}{89600}$ | $\frac{282843}{89600}$ | $-\frac{126927}{11200}$ | $\frac{38691}{1120}$ | $-\frac{2612817}{44800}$ | $\frac{3265839}{44800}$ | $-\frac{334233}{5600}$ | $\frac{407457}{11200}$ | $-\frac{1147365}{89600}$ | $\frac{398799}{89600}$ | $\frac{43928811}{44}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-9}^{1} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 10-strip | | | | | | |
| 54. | 10 | 0 | $\frac{40450}{9072}$ | $-\frac{29225}{2268}$ | $\frac{41675}{1134}$ | $-\frac{137675}{2268}$ | $\frac{169555}{2268}$ | $-\frac{137675}{2268}$ | $\frac{41675}{1134}$ | $-\frac{29225}{2268}$ | $\frac{40450}{9072}$ | $\frac{32134000}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-1}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1-strip | | | | | | |
| 55. | 2 | $\frac{1}{2}$ | $\frac{1}{2}$ | (Trapezoidal Rule) | | | | | | | | $-\frac{1}{6}$ |
| 56. | 3 | $-\frac{1}{12}$ | $\frac{2}{3}$ | $\frac{5}{12}$ | | | | | | | | $-\frac{1}{4}$ |
| 57. | 4 | $\frac{1}{24}$ | $-\frac{5}{24}$ | $\frac{19}{24}$ | $\frac{3}{8}$ | | | | | | | $-\frac{19}{30}$ |
| 58. | 5 | $-\frac{19}{720}$ | $\frac{53}{360}$ | $-\frac{11}{30}$ | $\frac{323}{360}$ | $\frac{251}{720}$ | | | | | | $-\frac{9}{4}$ |
| 59. | 6 | $\frac{3}{160}$ | $-\frac{173}{1440}$ | $\frac{241}{720}$ | $-\frac{133}{240}$ | $\frac{1427}{1440}$ | $\frac{95}{288}$ | | | | | $-\frac{863}{84}$ |
| 60. | 7 | $-\frac{863}{60480}$ | $\frac{263}{2520}$ | $-\frac{6737}{20160}$ | $\frac{586}{945}$ | $-\frac{15487}{20160}$ | $\frac{2713}{2520}$ | $\frac{19087}{60480}$ | | | | $-\frac{1375}{24}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-2}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 61. | 8 | $\frac{275}{24192}$ | $-\frac{11351}{120960}$ | $\frac{1537}{4480}$ | $-\frac{88547}{120960}$ | $\frac{123133}{120960}$ | $-\frac{4511}{4480}$ | $\frac{139849}{120960}$ | $\frac{5257}{17280}$ | | | $-\frac{33953}{90}$ |
| 62. | 9 | $-\frac{33953}{3628800}$ | $\frac{156437}{1814400}$ | $-\frac{645607}{1814400}$ | $\frac{1573169}{1814400}$ | $-\frac{31457}{22680}$ | $\frac{2797679}{1814400}$ | $-\frac{2302297}{1814400}$ | $\frac{2233547}{1814400}$ | $\frac{1070017}{3628800}$ | | $-\frac{57281}{20}$ |
| 63. | 10 | $\frac{8183}{1036800}$ | $-\frac{116687}{1451520}$ | $\frac{335983}{907200}$ | $-\frac{462127}{453600}$ | $\frac{6755041}{3628800}$ | $-\frac{8641823}{3628800}$ | $\frac{200029}{90720}$ | $-\frac{1408913}{907200}$ | $\frac{9449717}{7257600}$ | $\frac{25713}{89600}$ | $-\frac{3250433}{132}$ |

2-strip

(Simpson's $\frac{1}{3}$ Rule)

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-2}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 64. | 4 | 0 | $\frac{1}{3}$ | $\frac{4}{3}$ | $\frac{1}{3}$ | | | | | | | $-\frac{4}{15}$ |
| 65. | 5 | $-\frac{1}{90}$ | $\frac{2}{45}$ | $\frac{4}{15}$ | $\frac{62}{45}$ | $\frac{29}{90}$ | | | | | | $-\frac{4}{3}$ |
| 66. | 6 | $\frac{1}{90}$ | $-\frac{1}{15}$ | $\frac{7}{45}$ | $\frac{7}{45}$ | $\frac{43}{30}$ | $\frac{14}{45}$ | | | | | $-\frac{148}{21}$ |
| 67. | 7 | $-\frac{37}{3780}$ | $\frac{22}{315}$ | $-\frac{269}{1260}$ | $\frac{332}{945}$ | $\frac{11}{1260}$ | $\frac{94}{63}$ | $\frac{1139}{3780}$ | | | | $-\frac{128}{3}$ |
| 68. | 8 | $\frac{8}{945}$ | $-\frac{29}{420}$ | $\frac{26}{105}$ | $-\frac{1927}{3780}$ | $\frac{68}{105}$ | $-\frac{71}{420}$ | $\frac{1466}{945}$ | $\frac{41}{140}$ | | | $-\frac{13328}{45}$ |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 69. | 9 | $-\frac{119}{16200}$ | $-\frac{953}{14175}$ | $\frac{15577}{56700}$ | $\frac{9341}{14175}$ | $-\frac{2903}{2835}$ | $\frac{15011}{14175}$ | $-\frac{21247}{56700}$ | $\frac{22823}{14175}$ | $\frac{32377}{113400}$ | | $-\frac{11664}{5}$ |
| 70. | 10 | $\frac{9}{1400}$ | $-\frac{3697}{56700}$ | $\frac{8467}{28350}$ | $-\frac{9239}{11340}$ | $\frac{83291}{56700}$ | $-\frac{103987}{56700}$ | $\frac{45331}{28350}$ | $-\frac{34369}{56700}$ | $\frac{37829}{22680}$ | $\frac{3956}{14175}$ | $-\frac{680080}{33}$ |
| No. | n+1 | $C^*_0$ | $C^*_1$ | $C^*_2$ | $C^*_3$ | $C^*_4$ | $C^*_5$ | $C^*_6$ | $C^*_7$ | $C^*_8$ | $C^*_9$ | $\int_{-3}^{0} \varphi(u+n)du$ |
| | | | | | | 3–strip | | | | | | |
| 71. | 4 | $\frac{3}{8}$ | $\frac{9}{8}$ | $\frac{9}{8}$ | $\frac{3}{8}$ | (Simpson's $\frac{3}{8}$ Rule) | | | | | | $-\frac{9}{10}$ |
| 72. | 5 | $-\frac{3}{80}$ | $\frac{21}{40}$ | $\frac{9}{10}$ | $\frac{51}{40}$ | $\frac{27}{80}$ | | | | | | $-\frac{9}{4}$ |
| 73. | 6 | $\frac{3}{160}$ | $-\frac{21}{160}$ | $\frac{57}{80}$ | $\frac{57}{80}$ | $\frac{219}{160}$ | $\frac{51}{160}$ | | | | | $-\frac{261}{28}$ |
| 74. | 7 | $-\frac{29}{2240}$ | $\frac{27}{280}$ | $-\frac{729}{2240}$ | $\frac{34}{35}$ | $\frac{1161}{2240}$ | $\frac{81}{56}$ | $\frac{137}{448}$ | | | | $-\frac{405}{8}$ |
| 75. | 8 | $\frac{9}{896}$ | $-\frac{373}{4480}$ | $\frac{1377}{4480}$ | $-\frac{3033}{4480}$ | $\frac{5927}{4480}$ | $\frac{1377}{4480}$ | $\frac{1359}{896}$ | $\frac{265}{896}$ | | | $-\frac{3321}{10}$ |
| 76. | 9 | $-\frac{369}{44800}$ | $\frac{243}{3200}$ | $-\frac{7031}{22400}$ | $\frac{17217}{22400}$ | $-\frac{351}{280}$ | $\frac{39967}{22400}$ | $\frac{1719}{22400}$ | $\frac{35451}{22400}$ | $\frac{12881}{44800}$ | | $-\frac{10125}{4}$ |
| 77. | 10 | $\frac{75}{10752}$ | $-\frac{909}{12800}$ | $\frac{3663}{11200}$ | $-\frac{5039}{5600}$ | $\frac{73809}{44800}$ | $-\frac{19107}{8960}$ | $\frac{13273}{5600}$ | $-\frac{279}{1600}$ | $\frac{147429}{89600}$ | $\frac{75411}{268800}$ | $-\frac{963819}{44}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-4}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 4-strip | | | | | | |
| 78. | 6 | 0 | $\frac{14}{45}$ | $\frac{64}{45}$ | $\frac{8}{15}$ | $\frac{64}{45}$ | $\frac{14}{45}$ | | | | | $-\frac{128}{21}$ |
| 79. | 7 | $-\frac{8}{945}$ | $\frac{16}{315}$ | $\frac{58}{315}$ | $\frac{1504}{945}$ | $\frac{128}{315}$ | $\frac{464}{315}$ | $\frac{286}{945}$ | | | | $-\frac{128}{3}$ |
| 80. | 8 | $\frac{8}{945}$ | $-\frac{64}{945}$ | $\frac{8}{35}$ | $-\frac{106}{945}$ | $\frac{1784}{945}$ | $\frac{8}{35}$ | $\frac{1448}{945}$ | $\frac{278}{945}$ | | | $-\frac{13696}{45}$ |
| 81. | 9 | $-\frac{107}{14175}$ | $\frac{976}{14175}$ | $-\frac{3956}{14175}$ | $\frac{9232}{14175}$ | $-\frac{1816}{2835}$ | $\frac{32752}{14175}$ | $\frac{244}{14175}$ | $\frac{22576}{14175}$ | $\frac{4063}{14175}$ | | $-\frac{12032}{5}$ |
| 82. | 10 | $\frac{94}{14175}$ | $-\frac{953}{14175}$ | $\frac{872}{2835}$ | $-\frac{11852}{14175}$ | $\frac{21076}{14175}$ | $-\frac{20924}{14175}$ | $-\frac{40648}{14175}$ | $-\frac{3140}{14175}$ | $\frac{23422}{14175}$ | $\frac{7}{25}$ | $-\frac{700160}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-5}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 5-strip | | | | | | |
| 83. | 6 | $\frac{95}{288}$ | $\frac{125}{96}$ | $\frac{125}{144}$ | $\frac{125}{144}$ | $\frac{125}{96}$ | $\frac{95}{288}$ | | | | | $-\frac{1375}{84}$ |
| 84. | 7 | $-\frac{275}{12096}$ | $\frac{235}{504}$ | $\frac{3875}{4032}$ | $\frac{250}{189}$ | $\frac{2125}{4032}$ | $\frac{725}{504}$ | $\frac{3715}{12096}$ | | | | $-\frac{1375}{24}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-6}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 85. | 8 | $\frac{275}{24192}$ | $-\frac{275}{2688}$ | $\frac{1895}{2688}$ | $\frac{13625}{24192}$ | $\frac{4625}{2688}$ | $\frac{775}{2688}$ | $\frac{36725}{24192}$ | $\frac{265}{896}$ | | | $-\frac{6125}{18}$ |
| 86. | 9 | $-\frac{175}{20736}$ | $\frac{5725}{72576}$ | $-\frac{24575}{72576}$ | $\frac{85465}{72576}$ | $-\frac{125}{4536}$ | $\frac{159175}{72576}$ | $\frac{3775}{72576}$ | $\frac{115075}{72576}$ | $\frac{41705}{145152}$ | | $-\frac{10125}{4}$ |
| 87. | 10 | $\frac{25}{3584}$ | $-\frac{20675}{290304}$ | $\frac{11975}{36288}$ | $-\frac{16775}{18144}$ | $\frac{298505}{145152}$ | $-\frac{131575}{145152}$ | $\frac{50425}{18144}$ | $-\frac{7225}{36288}$ | $\frac{478525}{290304}$ | $\frac{81385}{290304}$ | $+\frac{2874625}{132}$ |

6-strip

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 88. | 8 | 0 | $\frac{41}{140}$ | $\frac{54}{35}$ | $\frac{27}{140}$ | $\frac{68}{35}$ | $\frac{27}{140}$ | $\frac{54}{35}$ | $\frac{41}{140}$ | | | $-\frac{1296}{5}$ |
| 89. | 9 | $-\frac{9}{1400}$ | $\frac{9}{175}$ | $\frac{79}{700}$ | $\frac{333}{175}$ | $-\frac{9}{35}$ | $\frac{403}{175}$ | $\frac{9}{700}$ | $\frac{279}{175}$ | $\frac{401}{1400}$ | | $-\frac{11664}{5}$ |
| 90. | 10 | $\frac{9}{1400}$ | $-\frac{9}{140}$ | $\frac{99}{350}$ | $-\frac{299}{700}$ | $\frac{1899}{700}$ | $-\frac{747}{700}$ | $\frac{199}{70}$ | $-\frac{153}{700}$ | $\frac{2313}{1400}$ | $\frac{49}{175}$ | $-\frac{231984}{11}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-7}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 7-strip | | | | | | |
| 91. | 8 | $\frac{5257}{17280}$ | $\frac{25039}{17280}$ | $\frac{9261}{17280}$ | $\frac{20923}{17280}$ | $\frac{20923}{17280}$ | $\frac{9261}{17280}$ | $\frac{25039}{17280}$ | $\frac{5257}{17280}$ | | | $-\frac{57281}{90}$ |
| 92. | 9 | $-\frac{8183}{518400}$ | $\frac{111587}{259200}$ | $\frac{261023}{259200}$ | $\frac{368039}{259200}$ | $\frac{343}{3240}$ | $\frac{542969}{259200}$ | $\frac{24353}{259200}$ | $\frac{408317}{259200}$ | $\frac{149527}{518400}$ | | $-\frac{57281}{20}$ |
| 93. | 10 | $\frac{8183}{1036800}$ | $-\frac{90013}{1036800}$ | $\frac{92617}{129600}$ | $\frac{4459}{12960}$ | $\frac{1251607}{518400}$ | $-\frac{460649}{518400}$ | $\frac{178703}{64800}$ | $-\frac{24647}{129600}$ | $\frac{1706915}{1036800}$ | $\frac{3591}{12800}$ | $-\frac{2954945}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-8}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 8-strip | | | | | | |
| 94. | 10 | 0 | $\frac{3956}{14175}$ | $\frac{23552}{14175}$ | $-\frac{3712}{14175}$ | $\frac{41984}{14175}$ | $-\frac{3632}{2835}$ | $\frac{41984}{14175}$ | $-\frac{3712}{14175}$ | $\frac{23552}{14175}$ | $\frac{3956}{14175}$ | $-\frac{606208}{33}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-9}^{0} \varphi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 9-strip | | | | | | |
| 95. | 10 | $\frac{25713}{89600}$ | $\frac{141669}{89600}$ | $\frac{243}{2240}$ | $\frac{10881}{5600}$ | $\frac{26001}{44800}$ | $\frac{26001}{44800}$ | $\frac{10881}{5600}$ | $\frac{243}{2240}$ | $\frac{141669}{89600}$ | $\frac{25713}{89600}$ | $-\frac{1891755}{44}$ |

The following formulae are self-explanatory.

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-2}^{-1} \phi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 96. | 4 | $-\frac{1}{24}$ | $\frac{13}{24}$ | $\frac{13}{24}$ | $-\frac{1}{24}$ | | | | | | | $\frac{11}{30}$ |
| 97. | 5 | $\frac{11}{720}$ | $-\frac{37}{360}$ | $\frac{19}{30}$ | $\frac{173}{360}$ | $-\frac{19}{720}$ | | | | | | $\frac{11}{12}$ |
| 98. | 6 | $-\frac{11}{1440}$ | $\frac{77}{1440}$ | $-\frac{43}{240}$ | $\frac{511}{720}$ | $\frac{637}{1440}$ | $-\frac{3}{160}$ | | | | | $\frac{271}{84}$ |
| 99. | 7 | $\frac{271}{60480}$ | $-\frac{29}{840}$ | $\frac{811}{6720}$ | $-\frac{254}{945}$ | $\frac{5221}{6720}$ | $\frac{349}{840}$ | $-\frac{863}{60480}$ | | | | $\frac{117}{8}$ |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100. | 8 | $-\frac{13}{4480}$ | $\frac{2999}{120960}$ | $-\frac{1283}{13440}$ | $\frac{2987}{13440}$ | $-\frac{44797}{120960}$ | $\frac{11261}{13440}$ | $\frac{5311}{13440}$ | $-\frac{275}{24192}$ | | | $\frac{7297}{90}$ |
| 101. | 9 | $\frac{7297}{3628800}$ | $-\frac{34453}{1814400}$ | $\frac{147143}{1814400}$ | $-\frac{377521}{1814400}$ | $\frac{8233}{22680}$ | $-\frac{876271}{1814400}$ | $\frac{1622393}{1814400}$ | $\frac{687797}{1814400}$ | $-\frac{33953}{3628800}$ | | $\frac{2125}{4}$ |
| 102. | 10 | $-\frac{425}{290304}$ | $\frac{110219}{7257600}$ | $-\frac{65039}{907200}$ | $\frac{92567}{453600}$ | $-\frac{1424417}{3628800}$ | $\frac{397331}{725760}$ | $-\frac{274849}{907200}$ | $\frac{859009}{907200}$ | $\frac{2655563}{7257600}$ | $\frac{8183}{1036800}$ | $\frac{530113}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-3}^{-2} \phi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 103. | 6 | $\frac{11}{1440}$ | $-\frac{31}{480}$ | $\frac{401}{720}$ | $\frac{401}{720}$ | $-\frac{31}{480}$ | $\frac{11}{1440}$ | | | | | $-\frac{191}{84}$ |
| 104 | 7 | $-\frac{191}{60480}$ | $\frac{67}{2520}$ | $-\frac{2257}{20160}$ | $\frac{586}{945}$ | $\frac{10273}{20160}$ | $-\frac{23}{504}$ | $\frac{271}{60480}$ | | | | $-\frac{191}{24}$ |
| 105. | 8 | $\frac{191}{120960}$ | $-\frac{191}{13440}$ | $\frac{803}{13440}$ | $-\frac{20227}{120960}$ | $\frac{9077}{13440}$ | $\frac{6403}{13440}$ | $-\frac{4183}{120960}$ | $\frac{13}{4480}$ | | | $-\frac{3233}{90}$ |
| 106. | 9 | $-\frac{3233}{3628800}$ | $\frac{14797}{1814400}$ | $\frac{71047}{1814400}$ | $\frac{198929}{1814400}$ | $-\frac{5207}{22680}$ | $\frac{1315919}{1814400}$ | $\frac{819143}{1814400}$ | $-\frac{49813}{1814400}$ | $\frac{7297}{3628800}$ | | $-\frac{3969}{20}$ |
| 107. | 10 | $\frac{7}{12800}$ | $-\frac{42187}{7257600}$ | $\frac{25759}{907200}$ | $-\frac{38599}{453600}$ | $\frac{129581}{725760}$ | $-\frac{1083167}{3628800}$ | $\frac{349817}{453600}$ | $\frac{391711}{907200}$ | $-\frac{163531}{7257600}$ | $\frac{425}{290304}$ | $-\frac{171137}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-4}^{-3} \phi(u+n)dx$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 108. | 7 | $\frac{271}{60480}$ | $-\frac{23}{504}$ | $\frac{10273}{20160}$ | $\frac{586}{2945}$ | $-\frac{2257}{20160}$ | $\frac{67}{2520}$ | $-\frac{191}{60480}$ | | | | $\frac{191}{24}$ |
| 109. | 8 | $-\frac{191}{120960}$ | $\frac{1879}{120960}$ | $-\frac{353}{4480}$ | $\frac{68323}{120960}$ | $\frac{68323}{120960}$ | $-\frac{353}{4480}$ | $\frac{1879}{120960}$ | $+\frac{191}{120960}$ | | | $\frac{2497}{90}$ |
| 110. | 9 | $\frac{2497}{3628800}$ | $-\frac{12853}{1814400}$ | $\frac{63143}{1814400}$ | $-\frac{212881}{1814400}$ | $\frac{13903}{22680}$ | $\frac{954929}{1814400}$ | $-\frac{108007}{1814400}$ | $\frac{18197}{1814400}$ | $-\frac{3233}{3628800}$ | | $\frac{2497}{20}$ |
| 111. | 10 | $-\frac{2497}{7257600}$ | $\frac{27467}{7257600}$ | $-\frac{17663}{907200}$ | $\frac{5779}{90720}$ | $-\frac{583073}{3628800}$ | $\frac{2381791}{3628800}$ | $\frac{225623}{453600}$ | $-\frac{42767}{907200}$ | $\frac{50315}{7257600}$ | $-\frac{21}{38400}$ | $\frac{90817}{132}$ |

| No. | n+1 | $C_0^*$ | $C_1^*$ | $C_2^*$ | $C_3^*$ | $C_4^*$ | $C_5^*$ | $C_6^*$ | $C_7^*$ | $C_8^*$ | $C_9^*$ | $\int_{-5}^{-4} \phi(u+n)du$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 112. | 8 | $\frac{13}{4480}$ | $-\frac{4183}{120960}$ | $\frac{6403}{13440}$ | $\frac{9077}{13440}$ | $-\frac{20227}{120960}$ | $\frac{803}{13440}$ | $-\frac{191}{13440}$ | $\frac{191}{12095}$ | | | $-\frac{3233}{90}$ |
| 113. | 9 | $-\frac{3233}{3628800}$ | $\frac{18197}{1814400}$ | $-\frac{108007}{1814400}$ | $\frac{954929}{1814400}$ | $\frac{13903}{22680}$ | $-\frac{212881}{1814400}$ | $\frac{63143}{1814400}$ | $-\frac{12853}{1814400}$ | $\frac{2497}{3628800}$ | | $-\frac{2497}{20}$ |
| 114. | 10 | $\frac{2497}{7257600}$ | $-\frac{28939}{7257600}$ | $\frac{581}{25920}$ | $-\frac{40111}{453600}$ | $\frac{2067169}{3628800}$ | $\frac{2067169}{3628800}$ | $-\frac{40111}{453600}$ | $\frac{581}{25920}$ | $-\frac{28939}{7257600}$ | $\frac{2497}{7257600}$ | $-\frac{73985}{132}$ |

## APPLICATION OF FORMULAE

Desirable closed formulae are easily seen to be No's 64, 78, 88, and 94. Some caution should be used when the 8-strip formula is employed for since some of its coefficients are negative the round-off error may appear excessive. Although some of the coefficients are negative, never-the-less, open formulae No's 27, 40, 49, and 54 show considerable appeal. For example No. 40 was employed in A.F. Problem No. 19 at Harvard Computation Laboratory.

Very little can be said in favor of increasing the error in order to simplify the coefficients when the numerical operations are performed on a large scale digital electronic calculator. Consequently Weddle's Rule, Hardy's Formula, Shovelton's Rule, etc. will not be introduced here.

Often it is necessary to change the mesh interval when integrating a system of differential equations. Formula (1) may be used to generate a set of formulae to cut such an interval. Obviously no special formulae are necessary to double an interval.

It is interesting to note that by decreasing the mesh size indefinitely and at the same time keeping all the ordinates (an infinite number in a finite interval) we are led to the cardinal interpolation function mentioned in AFTR No. 6581 by F. W. Bubb.

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